

# Entangling ability of a beam splitter in the presence of temporal which-path information

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We calculate the amount of polarization-entanglement induced by two-photon interference at a lossless beam splitter. Entanglement and its witness are quantified respectively by concurrence and the Bell-CHSH parameter. In the presence of a Mandel dip, the interplay of two kinds of which-path information — temporal and polarization — gives rise to the existence of entangled polarization-states that cannot violate the Bell-CHSH inequality.

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## I. INTRODUCTION

Entanglement, the nonclassical correlations between spatially separated particles, is typically a signature of interactions in the past or emergence from a common source. However, it can also arise as the interference of identical particles [1]. By postselecting experimental data based on the “click” of detectors [2, 3], photons scattered at a beam splitter have violated a Bell inequality, even if they originated from independent sources [4, 5]. In reverse, triggered by an interferometric Bell-state measurement, entanglement has been swapped [6] to initially uncorrelated photons of different Bell pairs [7, 8, 9]. The observation of these nonclassical interference effects is an important step on the road towards an optical approach of quantum information processing [10, 11].

Being furnished by interference, the ability of a beam splitter to entangle the polarizations of two independent photons depends on their indistinguishability [12]. One of the incident photons is horizontally polarized in state  $|H; \psi\rangle$ , the other vertically polarized in  $|V; \phi\rangle$ . The photons are partially distinguishable by their temporal degrees of freedom captured in the kets  $|\psi\rangle$  and  $|\phi\rangle$ . Besides temporal which-path information inherited from incident photons, a scattered two-photon state possibly holds polarization which-path information. We make no assumptions about the scattering amplitudes connecting polarizations at the beam splitter, except that they constitute a unitary scattering matrix. Translated to a polarization-conserving beam splitter, this corresponds to incident photons in states  $|\sigma; \psi\rangle$  and  $|\sigma'; \phi\rangle$  where  $\sigma, \sigma'$  are arbitrary superpositions of H, V. Our analysis generalizes existing work on a polarization-conserving beam splitter where  $\sigma = H$  and  $\sigma' = V$  [5, 13].

The polarization-state  $\rho$  of a scattered photon pair is established from the scattering amplitudes of the beam splitter, the shape and timing of photonic wavepackets ( $|\psi\rangle, |\phi\rangle$ ) and the time-window of coincidence detection. If not erased by ultra-coincidence detection, an amount

of temporal distinguishability of  $(1 - |\langle\psi|\phi\rangle|^2)$  pertains corresponding to a mixed state  $\rho$ . We calculate both its concurrence and the Bell-CHSH parameter. The ability of the latter to witness entanglement can disappear in the presence of a Mandel dip. In terms of a polarization-conserving beam splitter, this corresponds to a deviation of  $\sigma, \sigma'$  from  $\sigma = H$  and  $\sigma' = V$ .

## II. FORMULATION OF THE PROBLEM

In a second-quantized notation, the incident two-photon state  $|H; \psi\rangle_L |V; \phi\rangle_R$  takes the form

$$|\Psi_{\text{in}}\rangle = \Psi_{H,L}^\dagger \Phi_{V,R}^\dagger |0\rangle, \quad (1)$$

with field creation operators given by (see Fig. 1)

$$\Psi_{H,L}^\dagger = \int d\omega a_H^\dagger(\omega) \psi^*(\omega), \quad \Phi_{V,R}^\dagger = \int d\omega b_V^\dagger(\omega) \phi^*(\omega). \quad (2)$$

(The subscripts R,L indicate the two sides of the beam splitter.) The operators  $a_i(\omega)$  with  $i = H, V$  satisfy commutation rules

$$[a_i(\omega), a_j(\omega')] = 0, \quad [a_i(\omega), a_j^\dagger(\omega')] = \delta_{ij} \delta(\omega - \omega'). \quad (3)$$

The same commutation rules hold for the operators  $b_i(\omega)$ , with commutation among  $a$  and  $b$ .

The outgoing operators  $c_i(\omega), d_i(\omega)$  are related to the incoming ones  $a_i(\omega), b_i(\omega)$  by a  $4 \times 4$  unitary scattering matrix  $S$ , decomposed in  $2 \times 2$  reflection and transmission matrices  $r, t, t', r'$ :

$$\begin{pmatrix} c(\omega) \\ d(\omega) \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a(\omega) \\ b(\omega) \end{pmatrix}, \quad a(\omega) \equiv \begin{pmatrix} a_H(\omega) \\ a_V(\omega) \end{pmatrix}, \quad (4)$$

and vectors  $b(\omega), c(\omega), d(\omega)$  defined similarly. The scattering amplitudes are frequency-independent.

The outgoing state  $|\Psi_{\text{out}}\rangle$  can be conveniently written in a matrix notation

$$|\Psi_{\text{out}}\rangle = \int d\omega \int d\omega' \psi^*(\omega) \phi^*(\omega') \begin{pmatrix} c^\dagger(\omega) \\ d^\dagger(\omega) \end{pmatrix}^T \begin{pmatrix} r\sigma_{\text{in}} t'^T & r\sigma_{\text{in}} r'^T \\ t\sigma_{\text{in}} t'^T & t\sigma_{\text{in}} r'^T \end{pmatrix} \begin{pmatrix} c^\dagger(\omega') \\ d^\dagger(\omega') \end{pmatrix} |0\rangle. \quad (5)$$

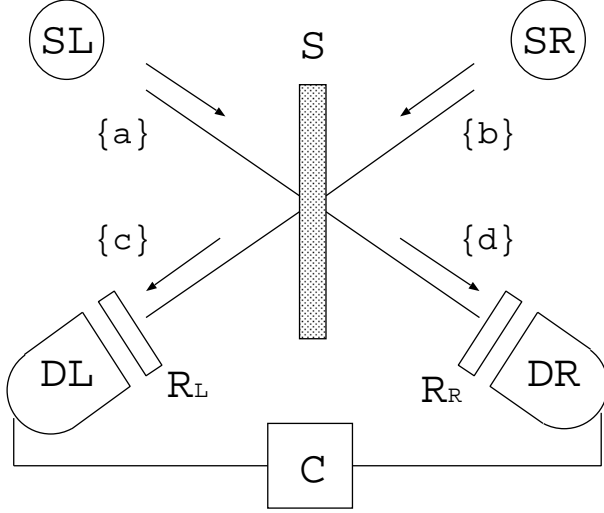


FIG. 1: Schematic drawing of generation and detection of polarization-entanglement at a beam splitter. The independent sources SL and SR each create a photon in modes  $\{a\}$  and  $\{b\}$  cf. Eq. (1). The beam splitter with unitary  $4 \times 4$  scattering matrix  $S$  couples the polarization of incoming modes to the polarization of outgoing modes  $\{c\}$  and  $\{d\}$ . Polarizations are mixed by  $R_L, R_R$  and absorbed by photodetectors DL, DR. A coincidence circuit C registers simultaneous detection of photons.

Here we used the unitarity of  $S$  and  $\sigma_{\text{in}} = (\sigma_x + i\sigma_y)/2$ , with  $\sigma_x$  and  $\sigma_y$  Pauli matrices, corresponds to the polarizations of the incoming photons cf. Eq. (1). The matrix  $\sigma_{\text{in}}$  has rank 1 reflecting the fact that polarizations are not entangled prior to scattering. Since we make no assumptions about the scattering amplitudes (apart from the unitarity of  $S$ ), the choice of  $\sigma_{\text{in}}$  is without loss of generality (see Appendix A).

The joint probability per unit (time)<sup>2</sup> of absorbing a photon with polarization  $i$  at detector DL and a pho-

ton with polarization  $j$  at detector DR at times  $t$  and  $t'$  respectively is given by [14]

$$w_{ij}(t, t') \propto \langle \Psi_{\text{out}} | E_{iL}^{(-)}(t) E_{jR}^{(-)}(t') E_{jR}^{(+)}(t') E_{iL}^{(+)}(t) | \Psi_{\text{out}} \rangle, \quad (6)$$

where  $E_{iL}^{(+)}(t)$  and  $E_{iR}^{(+)}(t)$  are the positive frequency field operators of polarization  $i$  at detectors DL and DR. The probability  $C_{ij}(t)$  of a coincidence event within time-windows  $\tau$  around  $t$  is given by

$$C_{ij}(t) = \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} dt' \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} dt'' w_{ij}(t', t''). \quad (7)$$

Experimentally, the time-window  $\tau$  has typically a lower bound determined by the random rise time of an avalanche of charge carriers in response to a photon absorption event.

The polarization-entanglement is detected by violation of the Bell-CHSH inequality [15]. This requires two local polarization mixers  $R_L$  and  $R_R$ . The Bell-CHSH parameter  $\mathcal{E}$  is

$$\mathcal{E} = |E(R_L, R_R) + E(R'_L, R_R) + E(R_L, R'_R) - E(R'_L, R'_R)|, \quad (8)$$

where  $E(R_L, R_R)$  is related to the correlators  $C_{ij}(R_L, R_R)$  by

$$E = \frac{C_{HH} + C_{VV} - C_{HV} - C_{VH}}{C_{HH} + C_{VV} + C_{HV} + C_{VH}}. \quad (9)$$

Substituting the correlators of Eq. (7) into Eq. (9), we see that

$$E(R_L, R_R) = \text{Tr} \rho (R_L^\dagger \sigma_z R_L) \otimes (R_R^\dagger \sigma_z R_R), \quad (10)$$

where  $\sigma_z$  is a Pauli matrix and  $\rho$  a  $4 \times 4$  polarization density matrix with elements

$$\rho_{ij,mn} = \frac{1}{\mathcal{N}} ((1 + |\alpha|^2)(\gamma_1)_{ij}(\gamma_1)_{mn}^* + (1 - |\alpha|^2)(\gamma_2)_{ij}(\gamma_2)_{mn}^*). \quad (11)$$

The parameter  $\alpha$  is given by

$$\alpha = \frac{\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} dt' \int d\omega \int d\omega' \phi(\omega) \psi^*(\omega') e^{i(\omega-\omega')t'}}{\sqrt{\left( \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} dt' \int d\omega \int d\omega' \phi(\omega) \phi^*(\omega') e^{i(\omega-\omega')t'} \right) \left( \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} dt' \int d\omega \int d\omega' \psi(\omega) \psi^*(\omega') e^{i(\omega-\omega')t'} \right)}} \quad (12)$$

and  $\gamma_1, \gamma_2$  are  $2 \times 2$  matrices related to the scattering

amplitudes by

$$\gamma_1 = r\sigma_{\text{in}} r'^T + t'\sigma_{\text{in}}^T t^T, \quad \gamma_2 = r\sigma_{\text{in}} r'^T - t'\sigma_{\text{in}}^T t^T. \quad (13)$$

The normalization factor  $\mathcal{N}$  takes the form

$$\mathcal{N} = (1 + |\alpha|^2) \text{Tr } \gamma_1^\dagger \gamma_1 + (1 - |\alpha|^2) \text{Tr } \gamma_2^\dagger \gamma_2. \quad (14)$$

The parameter  $1 - |\alpha|^2 \in (0, 1)$  represents the amount of temporal which-path information. Generally, the time-window  $\tau$  is much larger than the coherence times or temporal difference of the wavepackets. We may then take the limit  $\tau \rightarrow \infty$  and  $\alpha$  reduces to the overlap of wavepackets

$$\alpha = \int d\omega \phi(\omega) \psi^*(\omega). \quad (15)$$

In the opposite limit of ultra-coincidence detection where  $\tau \rightarrow 0$ , temporal which-path information is completely erased corresponding to  $|\alpha|^2 = 1$ .

### III. ENTANGLEMENT OF FORMATION

The entanglement of formation of the mixed state  $\rho$  is quantified by the concurrence  $\mathcal{C}$  [16] given by

$$\mathcal{C} = \max \left( 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right). \quad (16)$$

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The non-Hermitian matrix  $\rho\tilde{\rho}$  has eigenvalue-eigenvector decomposition

$$\rho\tilde{\rho} = \frac{|\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1|^2}{\mathcal{N}^2} \left( \sum_{i=1,2} \widehat{\gamma_i s_i} \right) \left( (1 + |\alpha|^2)^2 \widehat{s_1 s_1} + (1 - |\alpha|^2)^2 \widehat{s_2 s_2} \right) \left( \sum_{i=1,2} \widehat{\gamma_i s_i} \right)^{-1}, \quad (20)$$


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where we have defined orthonormal states  $s_1 = (1/2)(\mathbb{1} + \sigma_z)$  and  $s_2 = (1/2)(\sigma_x + i\sigma_y)$ . The pseudo-inverse is easily seen to be

$$\left( \sum_{i=1,2} \widehat{\gamma_i s_i} \right)^{-1} = \frac{1}{(\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1)^*} \left( \widehat{s_1 \tilde{\gamma}_1} - \widehat{s_2 \tilde{\gamma}_2} \right). \quad (21)$$

It follows that

$$\mathcal{C} = \frac{2|\alpha|^2 |\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1|}{\mathcal{N}}. \quad (22)$$

The trace that appears in the numerator of Eq. (22) is given by

$$|\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1| = 2\sqrt{\text{Det } X^\dagger X \text{Det}(\mathbb{1} - X^\dagger X)}, \quad (23)$$

where we have defined a “hybrid”  $2 \times 2$  matrix  $X$  as

$$X = \begin{pmatrix} r_{\text{HH}} & t'_{\text{HV}} \\ r_{\text{VH}} & t'_{\text{VV}} \end{pmatrix}. \quad (24)$$

The normalization factor  $\mathcal{N}$  given by Eq. (14) can be expressed in terms of  $X$  using

$$\text{Tr } \gamma_1^\dagger \gamma_1 = \text{Tr } X^\dagger X - 2 \text{Per } X^\dagger X, \quad (25)$$

The  $\lambda_i$ 's are the eigenvalues of the matrix product  $\rho\tilde{\rho}$ , where  $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ , in the order  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ . The concurrence ranges from 0 (no entanglement) to 1 (maximal entanglement). For simplicity of notation it is convenient to define  $(\widehat{xy})_{ij,mn} \equiv x_{ij} y_{mn}^*$ . The matrix  $\tilde{\rho}$  can be written as

$$\tilde{\rho} = \frac{1}{\mathcal{N}} \left( (1 + |\alpha|^2) \widehat{\gamma_1 \gamma_1} + (1 - |\alpha|^2) \widehat{\gamma_2 \gamma_2} \right), \quad (17)$$

with  $\tilde{\gamma} \equiv \sigma_y \gamma^* \sigma_y$ . The product  $\rho\tilde{\rho}$  takes the simple form

$$\rho\tilde{\rho} = \frac{\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1}{\mathcal{N}^2} \left( (1 + |\alpha|^2)^2 \widehat{\gamma_1 \gamma_1} - (1 - |\alpha|^2)^2 \widehat{\gamma_2 \gamma_2} \right), \quad (18)$$

where we have used the multiplication rule  $\widehat{xyvw} = (\text{Tr } y^\dagger v) \widehat{xw}$  and

$$\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1 = -\text{Tr } \gamma_2^\dagger \tilde{\gamma}_2, \quad \text{Tr } \gamma_1^\dagger \tilde{\gamma}_2 = \text{Tr } \gamma_2^\dagger \tilde{\gamma}_1 = 0. \quad (19)$$

The results for the tilde inner products of Eq. (19) hold since the photons are not polarization-entangled prior to scattering ( $\text{Det } \sigma_{\text{in}} = 0$ ).

$$\text{Tr } \gamma_2^\dagger \gamma_2 = \text{Tr } X^\dagger X - 2 \text{Det } X^\dagger X. \quad (26)$$

(“Per” denotes the permanent of a matrix.) In the derivation of Eqs. (23,25,26) we have made use of the unitarity of  $S$ . The concurrence becomes

$$\mathcal{C} = \frac{2|\alpha|^2 \sqrt{\text{Det } X^\dagger X \text{Det}(\mathbb{1} - X^\dagger X)}}{\text{Tr } X^\dagger X - (1 + |\alpha|^2) \text{Per } X^\dagger X - (1 - |\alpha|^2) \text{Det } X^\dagger X}. \quad (27)$$

Entanglement depends on the amount of temporal indistinguishability  $|\alpha|^2$  and the Hermitian matrix

$$X^\dagger X = \begin{pmatrix} |\mathbf{r}_\text{H}|^2 & \mathbf{r}_\text{H} \cdot \mathbf{t}'_\text{V} \\ (\mathbf{r}_\text{H} \cdot \mathbf{t}'_\text{V})^* & |\mathbf{t}'_\text{V}|^2 \end{pmatrix}, \quad (28)$$

containing the states  $\mathbf{r}_\text{H} = (r_{\text{HH}}, r_{\text{VH}})$  and  $\mathbf{t}'_\text{V} = (t'_{\text{HV}}, t'_{\text{VV}})$  of a reflected and transmitted photon to the left of the beam splitter. The determinant of  $X^\dagger X$  measures the size of the span of  $\mathbf{r}_\text{H}$  and  $\mathbf{t}'_\text{V}$  as

$$\text{Det } X^\dagger X = |\mathbf{r}_\text{H}|^2 |\mathbf{t}'_\text{V}|^2 \left( 1 - \frac{|\mathbf{r}_\text{H} \cdot \mathbf{t}'_\text{V}|^2}{|\mathbf{r}_\text{H}|^2 |\mathbf{t}'_\text{V}|^2} \right). \quad (29)$$

If  $\mathbf{r}_H$  and  $\mathbf{t}'_V$  are parallel ( $\text{Det} X^\dagger X = 0$ ), a scattered photon to the left of the beam splitter is in a definite state, giving rise to an unentangled two-photon state ( $\mathcal{C} = 0$ ). Similarly,

$$\text{Det}(\mathbb{1} - X^\dagger X) = |\mathbf{t}_H|^2 |\mathbf{r}'_V|^2 \left( 1 - \frac{|\mathbf{t}_H \cdot \mathbf{r}'_V|^2}{|\mathbf{t}_H|^2 |\mathbf{r}'_V|^2} \right) \quad (30)$$

involves scattered states  $\mathbf{t}_H = (t_{HH}, t_{VH})$  and  $\mathbf{r}'_V = (r'_{HV}, r'_{VV})$  to the right of the beam splitter. The denominator of Eq. (27) is the probability of finding a scattered state with one photon on either side of the beam splitter. It deviates from its classical value  $(X^\dagger X)_{HH} + (X^\dagger X)_{VV} - 2(X^\dagger X)_{HH}(X^\dagger X)_{VV}$  by an amount  $-2|\alpha|^2 |(X^\dagger X)_{HV}|^2$  due to photon bunching. This reduction of coincidence count probability is the Mandel dip [17]. It measures the indistinguishability of a reflected and transmitted photon as the product of temporal indistinguishability  $|\alpha|^2$  and polarization indistinguishability  $|(X^\dagger X)_{HV}|^2$ .

#### IV. VIOLATION OF THE BELL-CHSH INEQUALITY

The maximal value  $\mathcal{E}_{\max}$  of the Bell-CHSH parameter (8) for an arbitrary mixed state was analyzed in Refs. [18, 19]. For a pure state with concurrence  $\mathcal{C}$  one has simply  $\mathcal{E}_{\max} = 2\sqrt{1 + \mathcal{C}^2}$  [20]. For a mixed state there is no one-to-one relation between  $\mathcal{C}$  and  $\mathcal{E}_{\max}$ . Depending on the density matrix,  $\mathcal{E}_{\max}$  can take on values between  $2\mathcal{C}\sqrt{2}$  and  $2\sqrt{1 + \mathcal{C}^2}$ . The dependence of  $\mathcal{E}_{\max}$  on  $\rho$  involves the two largest eigenvalues of the real symmetric  $3 \times 3$  matrix  $R^T R$  constructed from  $R_{kl} = \text{Tr} \rho \sigma_k \otimes \sigma_l$ , where  $\sigma_1 = \sigma_x, \sigma_2 = \sigma_y$  and  $\sigma_3 = \sigma_z$ . In terms of  $\gamma_1$  and  $\gamma_2$ , the elements  $R_{kl}$  take the form

$$R_{kl} = \frac{(1 + |\alpha|^2)}{\mathcal{N}} \text{Tr} \gamma_1^\dagger \sigma_k \gamma_1 \sigma_l^T + \frac{(1 - |\alpha|^2)}{\mathcal{N}} \text{Tr} \gamma_2^\dagger \sigma_k \gamma_2 \sigma_l^T. \quad (31)$$

The matrix  $\gamma_2$  has a polar decomposition  $\gamma_2 = U\sqrt{\xi}V$  where  $U$  and  $V$  are unitary matrices and  $\xi$  is a diagonal matrix holding the eigenvalues of  $\gamma_2^\dagger \gamma_2$ . The real positive  $\xi_i$ 's are determined by

$$\xi_1 + \xi_2 = \text{Tr} \gamma_2^\dagger \gamma_2, \quad 2\sqrt{\xi_1 \xi_2} = |\text{Tr} \gamma_2^\dagger \tilde{\gamma}_2|. \quad (32)$$

The matrix  $\gamma_1$  can be conveniently expressed as (see Ap-

pendix B)

$$\gamma_1 = UQ\sqrt{\xi}V, \quad \text{where} \quad Q = \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}. \quad (33)$$

The parameters  $c_1, c_2, c_3$  are real numbers. The matrix  $Q$  is traceless due to the orthogonality of  $\gamma_1$  and  $\tilde{\gamma}_2$ . The number  $c_1 \in (-1, 1)$  on the diagonal is related to the inner product of  $\gamma_1$  and  $\gamma_2$  and takes the form

$$c_1 = \frac{\text{Tr} \gamma_1^\dagger \gamma_2}{\xi_1 - \xi_2}, \quad \text{with} \quad \text{Tr} \gamma_1^\dagger \gamma_2 = \text{Tr} \sigma_z X^\dagger X. \quad (34)$$

The numbers  $c_2, c_3$  are determined by the norm and tilde inner product of  $\gamma_1$  and satisfy the relations

$$c_1^2 + c_2 c_3 = 1, \quad c_1^2(\xi_1 + \xi_2) + c_2^2 \xi_2 + c_3^2 \xi_1 = \text{Tr} \gamma_1^\dagger \gamma_1. \quad (35)$$

We substitute  $\gamma_1$  of Eq. (33) and the polar decomposition of  $\gamma_2$  in Eq. (31) and parameterize

$$U^\dagger \sigma_k U = \sum_{i=1}^3 N_{ki} \sigma_i, \quad V \sigma_k^T V^\dagger = \sum_{i=1}^3 M_{ki} \sigma_i^T, \quad (36)$$

in terms of two  $3 \times 3$  orthogonal matrices  $N$  and  $M$ . The matrix  $R$  takes the form

$$R = N R' M^T, \quad (37)$$

where  $R'$  is given by Eq. (31) with substitutions  $R \rightarrow R'$ ,  $\gamma_2 \rightarrow \sqrt{\xi}$  and  $\gamma_1 \rightarrow Q\sqrt{\xi}$ . With the help of Eqs. (32,34,35), the eigenvalues  $u_i$  of  $R^T R$  can now be expressed as (see Appendix C)

$$u_1 = \frac{1}{2\mathcal{N}^2} \left( \mathcal{T} + \sqrt{\mathcal{T}^2 - 4\mathcal{D}} \right), \quad (38)$$

$$u_2 = \frac{1}{2\mathcal{N}^2} \left( \mathcal{T} - \sqrt{\mathcal{T}^2 - 4\mathcal{D}} \right), \quad (39)$$

$$u_3 = 4 \frac{|\alpha|^4 |\text{Tr} \gamma_1^\dagger \tilde{\gamma}_1|^2}{\mathcal{N}^2}, \quad (40)$$

where

$$\mathcal{T} = \mathcal{N}^2 + 4|\text{Tr} \gamma_1^\dagger \tilde{\gamma}_1|^2 - 4(1 - |\alpha|^4) \left( \text{Tr} \gamma_1^\dagger \gamma_1 \text{Tr} \gamma_2^\dagger \gamma_2 - \text{Tr}^2 \gamma_1^\dagger \gamma_2 \right), \quad (41)$$

$$\mathcal{D} = 4|\text{Tr} \gamma_1^\dagger \tilde{\gamma}_1|^2 \left( \mathcal{N}^2 - 4(1 - |\alpha|^4) \text{Tr} \gamma_1^\dagger \gamma_1 \text{Tr} \gamma_2^\dagger \gamma_2 \right). \quad (42)$$

We can relate the  $u_i$ 's to  $X^\dagger X$  and  $|\alpha|^2$  using Eqs.

(14,23,25,26,34). The parameter  $\mathcal{E}_{\max}$  depends on the

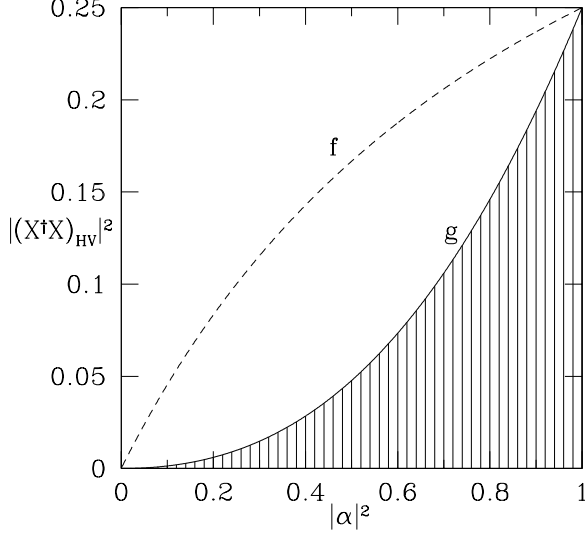


FIG. 2: Parameter space of a beam splitter with  $(X^\dagger X)_{ii} = 1/2$  spanned by  $|\alpha|^2 \in (0, 1)$  and  $|(X^\dagger X)_{HV}|^2 \in (0, 1/4)$ . All points correspond to a non-vanishing polarization-entanglement ( $\mathcal{C} > 0$ ) except the line segments  $|\alpha|^2 = 0$  and  $|(X^\dagger X)_{HV}|^2 = 1/4$  where entanglement vanishes ( $\mathcal{C} = 0$ ). Only in the shaded region, the Bell-CHSH parameter is able to detect entanglement ( $\mathcal{E}_{\max} > 2$ ). The lines correspond to the functions  $f, g$  of Eqs. (47,48) respectively.

two largest eigenvalues of  $R^T R$  as

$$\mathcal{E}_{\max} = 2\sqrt{u_1 + \max(u_2, u_3)}. \quad (43)$$

Generically, the expression for  $\mathcal{E}_{\max}$  takes a complicated form where ordering of  $u_2$  and  $u_3$  depends on  $X^\dagger X$  and  $|\alpha|^2$ .

## V. DISCUSSION

The objective of the discussion is to reveal the role played by the Mandel dip  $-2|\alpha|^2|(X^\dagger X)_{HV}|^2$  in the connection between  $\mathcal{C}$  and  $\mathcal{E}_{\max}$ .

We first consider the case  $|(X^\dagger X)_{HV}|^2 = 0$ . The concurrence of Eq. (27) reduces to

$$\mathcal{C} = \frac{2|\alpha|^2 \prod_{i=H,V} \sqrt{(X^\dagger X)_{ii}(1 - (X^\dagger X)_{ii})}}{(X^\dagger X)_{HH} + (X^\dagger X)_{VV} - 2(X^\dagger X)_{HH}(X^\dagger X)_{VV}}. \quad (44)$$

The maximal value of the Bell-CHSH parameter takes the form

$$\mathcal{E}_{\max} = 2\sqrt{1 + \mathcal{C}^2} \quad (45)$$

and  $\mathcal{C} > 0$  implies  $\mathcal{E}_{\max} > 2$ .

In the presence of a Mandel dip ( $|\alpha|^2|(X^\dagger X)_{HV}|^2 > 0$ ), the ability of  $\mathcal{E}$  to witness entanglement can disappear.

We consider the special case  $(X^\dagger X)_{ii} = 1/2$ . The concurrence of Eq. (27) reduces to

$$\mathcal{C} = \frac{|\alpha|^2 (1 - 4|(X^\dagger X)_{HV}|^2)}{1 - 4|\alpha|^2|(X^\dagger X)_{HV}|^2}. \quad (46)$$

To find  $\mathcal{E}_{\max}$  we have to consider the ordering of  $u_2$  and  $u_3$  which depends on  $|(X^\dagger X)_{HV}|^2$  and  $|\alpha|^2$ . The function

$$f(|\alpha|^2) = \frac{|\alpha|^2}{2(1 + |\alpha|^2)} \quad (47)$$

divides parameter space in the region  $|(X^\dagger X)_{HV}|^2 \leq f$  where  $\mathcal{E}_{\max} = 2\sqrt{u_1 + u_3}$  and the region  $|(X^\dagger X)_{HV}|^2 > f$  where  $\mathcal{E}_{\max} = 2\sqrt{u_1 + u_2}$ . The equation  $\mathcal{E}_{\max} = 2$  has a solution  $g(|\alpha|^2)$  for  $|(X^\dagger X)_{HV}|^2$  that lies in the region  $|(X^\dagger X)_{HV}|^2 \leq f$ . The function  $g$  takes the form

$$g(|\alpha|^2) = \frac{1}{4} \left( 1 - |\alpha|^2 + |\alpha|^4 - (1 - |\alpha|^2)\sqrt{1 + |\alpha|^4} \right) \quad (48)$$

and breaks parameter space in two fundamental regions: a region  $|(X^\dagger X)_{HV}|^2 < g$  where  $\mathcal{E}_{\max} > 2$  and a region  $|(X^\dagger X)_{HV}|^2 > g$  where  $\mathcal{E}_{\max} < 2$ . We have drawn these regions in Fig. 2. The maximal value of the Bell-CHSH parameter is given by

$$\mathcal{E}_{\max} = 2\mathcal{C}|\alpha|^{-2}\sqrt{1 + |\alpha|^4} \quad (49)$$

in the region  $|(X^\dagger X)_{HV}|^2 \leq f$ .

## VI. CONCLUSIONS

In summary, we have calculated the amount of polarization-entanglement (concurrence  $\mathcal{C}$ ) and its witness (maximal value of the Bell-CHSH parameter  $\mathcal{E}$ ) induced by two-photon interference at a lossless beam splitter. The ability of  $\mathcal{E}$  to witness entanglement ( $\mathcal{E}_{\max} > 2$ ) depends on the Mandel dip  $-2|\alpha|^2|(X^\dagger X)_{HV}|^2$ . In the absence of a Mandel dip,  $\mathcal{C} > 0$  implies  $\mathcal{E}_{\max} > 2$  cf. Eq. (45), whereas in its presence this is not necessarily true. In the latter case, as we have demonstrated in Sec. V with  $(X^\dagger X)_{ii} = 1/2$ , the witnessing ability of  $\mathcal{E}$  depends on the individual contributions of temporal ( $|\alpha|^2$ ) and polarization indistinguishability ( $|(X^\dagger X)_{HV}|^2$ ).

Our results can be applied to interference of other kinds of particles, getting entangled in some  $2 \otimes 2$  Hilbert space and being “marked” by an additional degree of freedom. However, determining the indistinguishability parameter  $|\alpha|^2$  requires careful analysis of the detection scheme. In case of fermions, the matrices  $\gamma_1$  and  $\gamma_2$  of Eq. (13) are to be interchanged. Systems without a time-reversal symmetry are captured by the analysis, as we did not make use of the symmetry of the scattering matrix.

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### APPENDIX A: ARBITRARINESS OF TWO-PHOTON INPUT STATE

The unitary scattering matrix has a polar decomposition

$$S = \begin{pmatrix} K' & 0 \\ 0 & L' \end{pmatrix} \begin{pmatrix} \sqrt{\mathbb{1}-T} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{\mathbb{1}-T} \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix}, \quad (\text{A1})$$

where  $K', L', K, L$  are  $2 \times 2$  unitary matrices and  $T = \text{diag}(T_H, T_V)$  is a matrix of transmission eigenvalues  $T_H, T_V \in (0, 1)$ . The outgoing state  $|\Psi_{\text{out}}\rangle$  is related to the  $4 \times 4$  matrix

$$S \begin{pmatrix} 0 & \sigma_{\text{in}} \\ 0 & 0 \end{pmatrix} S^T \quad (\text{A2})$$

cf. Eq. (5). By group decomposition  $K = K_1 K_2$  and  $L = L_1 L_2$ ,  $|\Psi_{\text{out}}\rangle$  is easily seen to correspond to  $K_2 \sigma_{\text{in}} L_2^T$  scattered by  $S$  of Eq. (A1) with substitutions  $K \rightarrow K_1$  and  $L \rightarrow L_1$ .

### APPENDIX B: JOINT SEMI-POLAR DECOMPOSITION

The matrices  $\gamma_1$  and  $\gamma_2$  have a decomposition

$$\gamma_1 = U \mathcal{A} V, \quad \gamma_2 = U \sqrt{\xi} V, \quad (\text{B1})$$

where  $U, V$  are unitary matrices and  $\xi$  is a diagonal matrix of eigenvalues of  $\gamma_2^\dagger \gamma_2$ . As we do not yet specify  $\mathcal{A}$ , such a joint decomposition always exists. In our case, the matrices  $\gamma_1$  and  $\gamma_2$  have the special properties

$$\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1 = -\text{Tr } \gamma_2^\dagger \tilde{\gamma}_2, \quad \text{Tr } \gamma_1^\dagger \tilde{\gamma}_2 = 0, \quad (\text{B2})$$

$$|\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1| = 2\sqrt{\text{Det } X^\dagger X \text{Det}(\mathbb{1} - X^\dagger X)}, \quad (\text{B3})$$

$$\text{Tr } \gamma_1^\dagger \gamma_1 = \text{Tr } X^\dagger X - 2 \text{Per } X^\dagger X, \quad (\text{B4})$$

$$\text{Tr } \gamma_2^\dagger \gamma_2 = \text{Tr } X^\dagger X - 2 \text{Det } X^\dagger X, \quad (\text{B5})$$

$$\text{Tr } \gamma_1^\dagger \gamma_2 = \text{Tr } \sigma_z X^\dagger X. \quad (\text{B6})$$

It is the purpose of this appendix to demonstrate that  $\mathcal{A} = Q\sqrt{\xi}$  where  $Q$  is a real traceless matrix of Eq. (33) with  $c_1$  given by Eq. (34) and  $c_2, c_3$  satisfying Eq. (35).

The inner and tilde inner product of  $\gamma_1$  and  $\gamma_2$  take the form

$$\text{Tr } \gamma_1^\dagger \gamma_2 = \text{Tr } \mathcal{A}^\dagger \sqrt{\xi}, \quad (\text{B7})$$

$$\text{Tr } \gamma_1^\dagger \tilde{\gamma}_2 = (\text{Det } UV)^{-2} \text{Tr } \mathcal{A}^\dagger \sigma_y \sqrt{\xi} \sigma_y = 0. \quad (\text{B8})$$

(Here we have used the identity  $U \sigma_y U^T = \text{Det}^2 U \sigma_y$ , valid for any  $2 \times 2$  unitary matrix  $U$ .) The conditions of Eqs. (B7, B8) involve the diagonal elements of  $\mathcal{A}$  as respectively

$$\text{Tr } \gamma_1^\dagger \gamma_2 = \sqrt{\xi_1} \mathcal{A}_{11}^* + \sqrt{\xi_2} \mathcal{A}_{22}^*, \quad (\text{B9})$$

$$\text{Tr } \gamma_1^\dagger \tilde{\gamma}_2 = (\text{Det } UV)^{-2} \left( \sqrt{\xi_1} \mathcal{A}_{22}^* + \sqrt{\xi_2} \mathcal{A}_{11}^* \right) = 0. \quad (\text{B10})$$

It follows that  $\mathcal{A}_{11} = c_1^* \sqrt{\xi_1}$  and  $\mathcal{A}_{22} = -c_1^* \sqrt{\xi_2}$  where  $c_1$  is given by

$$c_1 = \frac{\text{Tr } \gamma_1^\dagger \gamma_2}{\xi_1 - \xi_2}. \quad (\text{B11})$$

The number  $c_1$  is real since  $\text{Tr } \gamma_1^\dagger \gamma_2 = \text{Tr } \sigma_z X^\dagger X \in \mathbb{R}$ .

The determinant of  $\mathcal{A}$  is fixed by  $\text{Tr } \gamma_1^\dagger \tilde{\gamma}_1 = -\text{Tr } \gamma_2^\dagger \tilde{\gamma}_2$  implying

$$\text{Det } \mathcal{A} = -\sqrt{\xi_1 \xi_2}. \quad (\text{B12})$$

It follows that  $\mathcal{A}_{12} = \mathcal{A}'_{12} e^{i\phi}$  and  $\mathcal{A}_{21} = \mathcal{A}'_{21} e^{-i\phi}$  with real  $\mathcal{A}'_{12}, \mathcal{A}'_{21}, \phi$ . The numbers  $\mathcal{A}'_{12}, \mathcal{A}'_{21}$  satisfy

$$c_1^2 \sqrt{\xi_1 \xi_2} + \mathcal{A}'_{12} \mathcal{A}'_{21} = \sqrt{\xi_1 \xi_2}, \quad (\text{B13})$$

$$c_1^2 (\xi_1 + \xi_2) + \mathcal{A}'_{12}^2 + \mathcal{A}'_{21}^2 = \text{Tr } \gamma_1^\dagger \gamma_1, \quad (\text{B14})$$

where Eq. (B14) comes from  $\text{Tr } \mathcal{A}^\dagger \mathcal{A} = \text{Tr } \gamma_1^\dagger \gamma_1$ . The undetermined phase  $\phi$  can be taken out,

$$\mathcal{A} = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}'_{12} \\ \mathcal{A}'_{21} & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}, \quad (\text{B15})$$

and absorbed in the unitary matrices  $U$  and  $V$  by the transformations

$$U \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \rightarrow U, \quad \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} V \rightarrow V. \quad (\text{B16})$$

(Note that these transformations also hold for  $\gamma_2$  since  $\sqrt{\xi}$  commutes with a diagonal matrix of phase factors.)

The matrix  $\mathcal{A}$  is related to  $Q$  by  $\mathcal{A} = Q\sqrt{\xi}$ . It is now easily seen that the matrix  $Q$  is real and traceless and takes the form of Eq. (33), with  $c_1$  given by Eq. (34) and  $c_2, c_3$  satisfying Eq. (35).

As a last step we perform a consistency check to demonstrate that Eqs. (B13,B14) have solutions for  $\mathcal{A}'_{12}$  and  $\mathcal{A}'_{21}$ . The Hermitian matrix  $X^\dagger X$  has an eigenvalue-eigenvector decomposition

$$X^\dagger X = W^\dagger \Lambda W. \quad (\text{B17})$$

In terms of the eigenvalues  $\Lambda_i \in (0,1)$  and the unitary matrix  $W$ , the inner product of  $\gamma_1$  and  $\gamma_2$  and the  $\xi_i$ 's take the form

$$\text{Tr } \gamma_1^\dagger \gamma_2 = \Lambda_1(|W_{11}|^2 - |W_{12}|^2) + \Lambda_2(|W_{21}|^2 - |W_{22}|^2), \quad (\text{B18})$$

$$\xi_1 = \Lambda_1(1 - \Lambda_2), \quad \xi_2 = \Lambda_2(1 - \Lambda_1). \quad (\text{B19})$$

It follows that  $c_1 = \cos 2\eta$ , where we have set  $|W_{11}| = |W_{22}| = \cos \eta$  and  $|W_{12}| = |W_{21}| = \sin \eta$ . Eqs. (B13,B14) can be expressed as respectively

$$\mathcal{A}'_{12}\mathcal{A}'_{21} = \sin^2 2\eta \sqrt{\Lambda_1\Lambda_2(1 - \Lambda_1)(1 - \Lambda_2)}, \quad (\text{B20})$$

$$\mathcal{A}'_{12}^2 + \mathcal{A}'_{21}^2 = \sin^2 2\eta (\Lambda_1(1 - \Lambda_1) + \Lambda_2(1 - \Lambda_2)). \quad (\text{B21})$$

Since

$$2\sqrt{\Lambda_1\Lambda_2(1 - \Lambda_1)(1 - \Lambda_2)} \leq \Lambda_1(1 - \Lambda_1) + \Lambda_2(1 - \Lambda_2) \quad (\text{B22})$$

a family of solutions exists.

### APPENDIX C: EIGENVALUES OF $R^T R$

The non-vanishing elements of  $R'$  are given by

$$R'_{11} = \frac{2}{\mathcal{N}} (1 - |\alpha|^2 - (1 + |\alpha|^2)(c_1^2 - c_2c_3)) \sqrt{\xi_1\xi_2}, \quad (\text{C1})$$

$$R'_{13} = \frac{2}{\mathcal{N}} (1 + |\alpha|^2)c_1(c_2\xi_2 + c_3\xi_1), \quad (\text{C2})$$

$$R'_{22} = \frac{2}{\mathcal{N}} (-1 + |\alpha|^2 + (1 + |\alpha|^2)(c_1^2 + c_2c_3)) \sqrt{\xi_1\xi_2}, \quad (\text{C3})$$

$$R'_{31} = \frac{2}{\mathcal{N}} (1 + |\alpha|^2)c_1(c_2 + c_3)\sqrt{\xi_1\xi_2}, \quad (\text{C4})$$

$$R'_{33} = \frac{1}{\mathcal{N}} ((1 - |\alpha|^2) + (1 + |\alpha|^2)c_1^2) (\xi_1 + \xi_2) \quad (\text{C5})$$

$$- \frac{1}{\mathcal{N}} (1 + |\alpha|^2)(c_2^2\xi_2 + c_3^2\xi_1). \quad (\text{C6})$$

The matrix  $R'^T R'$  has eigenvalues

$$u_1 = \frac{1}{2\mathcal{N}^2} (\mathcal{T} + \sqrt{\mathcal{T}^2 - 4\mathcal{D}}), \quad (\text{C7})$$

$$u_2 = \frac{1}{2\mathcal{N}^2} (\mathcal{T} - \sqrt{\mathcal{T}^2 - 4\mathcal{D}}), \quad (\text{C8})$$

$$u_3 = R'_{22}^2, \quad (\text{C9})$$

where  $\mathcal{T}, \mathcal{D}$  are the trace, determinant respectively of the  $2 \times 2$  real symmetric matrix

$$\mathcal{N}^2 \begin{pmatrix} R'_{11}^2 + R'_{31}^2 & R'_{11}R'_{13} + R'_{31}R'_{33} \\ R'_{11}R'_{13} + R'_{31}R'_{33} & R'_{13}^2 + R'_{33}^2 \end{pmatrix}. \quad (\text{C10})$$

By making use of Eqs. (34,35)  $u_3, \mathcal{T}, \mathcal{D}$  can be simplified to yield the results of Eqs. (40,41,42) respectively.

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